

# ON SOME INEQUALITIES FOR THE IDENTRIC, LOGARITHMIC AND RELATED MEANS

JÓZSEF SÁNDOR AND BARKAT ALI BHAYO

**ABSTRACT.** We offer new proofs, refinements as well as new results related to classical means of two variables, including the identric and logarithmic means.

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## 1. Introduction

Since last few decades, the inequalities involving the classical means such as arithmetic mean  $A$ , geometric mean  $G$ , identric mean  $I$  and logarithmic mean  $L$  and weighted geometric mean  $S$  have been studied extensively by numerous authors, e.g. see [1, 2, 4, 7, 8, 15, 16, 17].

For two positive real numbers  $a$  and  $b$ , we define

$$A = A(a, b) = \frac{a+b}{2}, \quad G = G(a, b) = \sqrt{ab},$$

$$L = L(a, b) = \frac{a-b}{\log(a) - \log(b)}, \quad a \neq b,$$

$$I = I(a, b) = \frac{1}{e} \left( \frac{a^a}{b^b} \right)^{1/(a-b)}, \quad a \neq b,$$

$$S = S(a, b) = (a^a b^b)^{1/(a+b)}.$$

For the historical background of these means we refer the reader to [2, 4, 5, 12, 15, 16, 17]. Generalizations, or related means are studied in [3, 8, 7, 10, 12, 14, 18]. Connections of these means with trigonometric or hyperbolic inequalities are pointed out in [3, 13, 6, 14, 17].

Our main result reads as follows:

**1.1. Theorem.** *For all distinct positive real numbers  $a$  and  $b$ , we have*

$$(1.2) \quad 1 < \frac{I}{\sqrt{I(A^2, G^2)}} < \frac{2}{\sqrt{e}}.$$

*Both bounds are sharp.*

**1.3. Theorem.** *For all distinct positive real numbers  $a$  and  $b$ , we have*

$$(1.4) \quad 1 < \frac{2I^2}{A^2 + G^2} < c$$

where  $c = 1.14\dots$ . The bounds are best possible.

**1.5. Remark. A.** The left side of (1.4) may be rewritten also as

$$(1.6) \quad I > Q(A, G),$$

where  $Q(x, y) = \sqrt{(x^2 + y^2)/2}$  denotes the root square mean of  $x$  and  $y$ . In 1995, Seiffert [25] proved the first inequality in (1.2) by using series representations, which is rather strong. Now we prove that, (1.6) is a refinement of the first inequality in (1.2). Indeed, by the known relation  $I(x, y) < A(x, y) = (x + y)/2$ , we can write

$$I(A^2, G^2) < (A^2 + G^2)/2 = Q(A, G)^2,$$

so one has:

$$(1.7) \quad I > Q(A, G) > \sqrt{I(A^2, G^2)}.$$

As we have  $I(x^2, y^2) > I(x, y)^2$  (see Sándor [15]), hence (1.7) offers also a refinement of

$$(1.8) \quad I > I(A, G).$$

Other refinements of (1.8) have been provided in a paper by Neuman and Sándor [10]. Similar inequalities involving the logarithmic mean, as well as Sándor's means  $X$  and  $Y$ , we quote [3, 13, 14]. In the second part of paper, similar results will be proved.

**B.** In 1991, Sándor [16] proved the inequality

$$(1.9) \quad I > (2A + G)/3.$$

It is easy to see that, the left side of (1.4) and (1.9) cannot be compared.

In 2001 Sándor and Trif [21] have proved the following inequality:

$$(1.10) \quad I^2 < (2A^2 + G^2)/3.$$

The left side of (1.4) offers a good companion to (1.10). We note that the inequality (1.10) and the right side of (1.4) cannot be compared.

In [25], Seiffert proved the following relation:

$$(1.11) \quad L(A^2, G^2) > L^2,$$

which was refined by Neuman and Sándor [10] (for another proof, see [8]) as follows:

$$(1.12) \quad L(A, G) > L.$$

We will prove with a new method the following refinement of (1.11) and a counterpart of (1.12):

**1.13. Theorem.** *We have*

$$(1.14) \quad L(A^2, G^2) = \frac{(A+G)}{2} L(A, G) > \frac{(A+G)}{2} L > L^2,$$

$$(1.15) \quad L(I, G) < L,$$

$$(1.16) \quad L < L(I, L) < L \cdot (I - L)/(L - G).$$

**1.17. Corollary.** *One has*

$$(1.18) \quad G \cdot I/L < \sqrt{I \cdot G} < L(I, G) < L,$$

$$(1.19) \quad (L(I, G))^2 < L \cdot L(I, G) < L(I^2, G^2) < L \cdot (I + G)/2.$$

**1.20. Remark. A.** Relation (1.18) improves the inequality

$$G \cdot I/L < L(I, G),$$

due to Neuman and Sándor [10]. Other refinements of the inequality

$$(1.21) \quad L < (I + G)/2$$

are provided in [19].

**B.** Relation (1.16) is indeed a refinement of (1.21), as the weaker inequality can be written as  $(I - L)/(L - G) > 1$ , which is in fact (1.21).

The mean  $S$  is strongly related to other classical means. For example, in 1993 Sándor [17] discovered the identity

$$(1.22) \quad S(a, b) = I(a^2, b^2)/I(a, b),$$

where  $I$  is the identric mean. Inequalities for the mean  $S$  may be found in [15, 17, 20].

The following result shows that  $I$  and  $S(A, G)$  cannot be compared, but this is not true in case of  $I$  and  $S(Q, G)$ . Even a stronger result holds true.

**1.23. Theorem.** *None of the inequalities  $I > S(A, G)$  or  $I < S(A, G)$  holds true. On the other hand, one has*

$$(1.24) \quad S(Q, G) > A > I,$$

$$(1.25) \quad I(Q, G) < A.$$

**1.26. Remark.** By (1.24) and (1.25), one could ask if  $I$  and  $I(Q, G)$  may be compared to each other. It is not difficult to see that, this becomes equivalent to one of the inequalities

$$(1.27) \quad \frac{y \log y}{y - 1} < (\text{or } >) \frac{x}{\tanh(x)}, \quad x > 0,$$

where  $y = \sqrt{\cosh(2x)}$ . By using the Mathematica Software [11], we can show that (1.27) with “ $<$ ” is not true for  $x = 3/2$ , while (1.27) with “ $>$ ” is not true for  $x = 2$ .

## 2. Lemmas and proofs of the main results

The following lemma will be utilized in our proofs.

**2.1. Lemma.** *For  $b > a > 0$  there exists an  $x > 0$  such that*

$$(2.2) \quad \frac{A}{G} = \cosh(x), \quad \frac{I}{G} = e^{x/\tanh(x)-1}.$$

*Proof.* For any  $a > b > 0$ , one can find an  $x > 0$  such that  $a = e^x \cdot G$  and  $b = e^{-x} \cdot G$ . Indeed, it is immediate that such an  $x$  is (by considering  $a/b = e^{2x}$ ),  $x = (1/2) \log(a/b) > 0$ . Now, as  $A = G \cdot (e^x + e^{-x})/2 = G \cosh(x)$ , we get  $A/G = \cosh(x)$ . Similarly, we get

$$I = G \cdot (1/e) \exp(x(e^x + e^{-x})/(e^x - e^{-x})),$$

which gives  $I/G = e^{x/\tanh(x)-1}$ . □

**Proof of Theorem 1.1.** For  $x > 0$ , we have  $I/G = e^{x/\tanh(x)-1}$  and  $A/G = \cosh(x)$  by Lemma 2.1. Since

$$\log(I(a, b)) = \frac{a \log a - b \log b}{a - b} - 1,$$

we get

$$\log(\sqrt{I((A/G)^2, 1)}) = \frac{\cosh(x)^2 \log(\cosh(x))}{\cosh(x)^2 - 1} - \frac{1}{2}.$$

By using this identity, and taking the logarithms in the second identity of (2.2), the inequality

$$0 < \log(I/G) - \log(\sqrt{I(A/G)^2, 1}) < \log 2 - 1/2$$

becomes

$$(2.3) \quad \frac{1}{2} < f(x) < \log 2,$$

where

$$f(x) = \frac{x}{\tanh(x)} - \frac{\log(\cosh(x))}{\tanh(x)^2}.$$

A simple computation (which we omit here) for the derivative of  $f(x)$  gives:

$$(2.4) \quad \sinh(x)^3 f'(x) = 2 \cosh(x) \log(\cosh(x)) - x \sinh(x).$$

The following inequality appears in [6]:

$$(2.5) \quad \log(\cosh(x)) > \frac{x}{2} \tanh(x), \quad x > 0,$$

which gives  $f'(x) > 0$ , so  $f(x)$  is strictly increasing in  $(0, \infty)$ . As  $\lim_{x \rightarrow 0} f(x) = 1/2$ , and  $\lim_{x \rightarrow \infty} f(x) = \log 2$ , the double inequality (2.3) follows. So we have obtained a new proof of (1.2). □

We note that Seiffert's proof is based on certain infinite series representations. Also, our proof shows that the constants 1 and  $2/\sqrt{e}$  in (1.2) are optimal.

**2.6. Lemma.** *Let*

$$f(x) = \frac{2x}{\tanh(x)} - \log\left(\frac{\cosh(x)^2 + 1}{2}\right), \quad x > 0.$$

*Then*

$$(2.7) \quad 2 < f(x) < f(1.606\dots) = 2.1312\dots$$

*Proof.* One has  $(\cosh(x)^2 + 1)/2f'(x) = g(x)$ , where

$$\begin{aligned} g(x) &= \sinh(x) \cosh(x)^3 x \cosh(x)^2 + \sinh(x) \cosh(x) - x \\ &\quad - \cosh(x) \sinh(x)^3 2 \sinh(x) \cosh(x) - x \cosh(x)^2 x, \end{aligned}$$

by remarking that

$$\sinh(x) \cosh(x)^3 - \cosh(x) \sinh(x)^3 = \sinh(x) \cosh(x).$$

Now, a simple computation gives

$$g'(x) = \sinh(x) \cdot (3 \sinh(x) - 2x \cosh(x)) = 3 \sinh(x) \cosh(x) \cdot k(x),$$

where  $k(x) = \tanh(x) - 2x/3$ . As it is well known that the function  $\tanh(x)/x$  is strictly decreasing, the equation  $\tanh(x)/x = 2/3$  can have at most a single solution. As  $\tanh(1) = 0.7615\dots > 2/3$  and  $\tanh(3/2) = 0.9051\dots < 1 = (2/3) \cdot (3/2)$ , we find that the equation  $k(x) = 0$  has a single solution  $x_0$  in  $(1, 3/2)$ , and also that  $k(x) > 0$  for  $x$  in  $(0, x_0)$  and  $k(x) < 0$  in  $(x_0, 3/2)$ . This means that the function  $g(x)$  is strictly increasing in the interval  $(0, x_0)$  and strictly decreasing in  $(x_0, \infty)$ . As  $g(1) = 0.24 > 0$ , clearly  $g(x_0) > 0$ , while  $g(2) = -3.01\dots < 0$  implies that there exists a single zero  $x_1$  of  $g(x)$  in  $(x_0, 2)$ . In fact, as  $g(3/2) = 0.21 > 0$ , we get that  $x_1$  is in  $(3/2, 2)$ .

From the above consideration we conclude that  $g(x) > 0$  for  $x \in (0, x_1)$  and  $g(x) < 0$  for  $x \in (x_1, \infty)$ . Therefore, the point  $x_1$  is a maximum point to the function  $f(x)$ . It is immediate that  $\lim_{x \rightarrow 0} f(x) = 2$ . On the other hand, we shall compute the limit of  $f(x)$  at  $\infty$ . Clearly  $t = \cosh(x)$  tends to  $\infty$  as  $x$  tends to  $\infty$ . Since  $\log(t^2 + 1) - \log(t^2) = \log((t^2 + 1)/t^2)$  tends to  $\log 1 = 0$ , we have to compute the limit of  $l(x) = 2x \cosh(x)/\sinh(x) - 2 \log(\cosh(x)) + \log 2$ . Here

$$2x \frac{\cosh(x)}{\sinh(x)} - 2 \log(\cosh(x)) = 2 \log\left(\frac{\exp(x \cosh(x)/\sinh(x))}{\cosh x}\right).$$

Now remark that  $(x \cosh(x) - x \sinh(x))/\sinh(x)$  tends to zero, as  $x \cosh(x) - x \sinh(x) = x \exp(-x)$ . As  $\exp(x)/\cosh x$  tends to 2, by the above remarks we get that the limit of  $l(x)$  is  $2 \log 2 + \log 2 = 3 \log 2 > 2$ . Therefore, the left side of inequality (2.7) is proved. The right side follows by the fact that  $f(x) < f(x_1)$ . By Mathematica Software<sup>®</sup> [11], we can find  $x_1 = 1.606\dots$  and  $f(x_1) = 2.1312\dots$   $\square$

**Proof of Theorem 1.3.** By Lemma 2.1, one has  $(I/G)^2 = \exp(2(x/\tanh(x) - 1))$ , while  $(A/G)^2 = \cosh(x)^2$ ,  $x > 0$ . It is immediate that, the left side of (2.7) implies the left side of (1.4). Now, by the right side of (2.7) one has  $I^2 < \exp(c_1)(A^2 + G^2)/2$ ,

where  $c_1 = f(x_1) - 2 = 0.13 \dots$ . Since  $\exp(0.13 \dots) = 1.14$ , we get also the right side of (1.4).  $\square$

**Proof of Theorem 1.13.** The first relation of (1.14) follows from the identity

$$L(x^2, y^2) = ((x + y)/2) \cdot L(x, y),$$

which is a consequence of the definition of logarithmic mean, by letting  $x = A, y = G$ . The second inequality of (1.14) follows by (1.12), while the third one is a consequence of the known inequality

$$(2.8) \quad L < (A + G)/2.$$

A simple proof of (2.8) can be found in [12]. For (1.15), by the definition of logarithmic mean, one has

$$L(I, G) = (I - G)/\log(I/G),$$

and on base of the known identity

$$\log(I/G) = A/L - 1$$

(see [15, 22]), we get

$$L(I, G) = ((I - G)/(A - L))L < L,$$

since the inequality  $(I - G)/(A - L) < 1$  can be rewritten as

$$I + L < A + G$$

due to Alzer (see [15]).

The first inequality of (1.16) follows by the fact that  $L$  is a mean (i.e. if  $x < y$  then  $x < L(x, y) < y$ ), and the well known relation  $L < I$  (see [15]) For the proof of last relation of (1.16) we will use a known inequality of Sándor ([15]), namely:

$$(2.9) \quad \log(I/L) > 1 - G/L.$$

Write now that  $L(I, L) = (I - L)/\log(I/L)$ , and apply (2.9). Therefore, the proof of (1.16) is finished.  $\square$

**Proof of Corollary 1.17.** The first inequality of (1.18) follows by the well known relation  $L > \sqrt{GI}$  (see [2]), while the second relation is a consequence of the classical relation  $L(x, y) > G(x, y)$  (see e.g. [15]) applied to  $x = I, y = G$ . The last relation is inequality (1.14).

The first inequality of (1.19) follows by (1.14), while the second one by  $L(I^2, G^2) = L(I, G) \cdot (I + G)/2$  and inequality  $L < (I + G)/2$ . The last inequality follows in the same manner.  $\square$

**Proof of Theorem 1.23.** Since the mean  $S$  is homogeneous, the relation  $I > S(A, G)$  may be rewritten as  $I/G > S(A/G, 1)$ , so by using logarithm and applying Lemma 2.1, this inequality may be rewritten as

$$(2.10) \quad \frac{x}{\tanh(x)} - 1 > \frac{\cosh(x) \log(\cosh(x))}{1 + \cosh(x)}, \quad x > 0.$$

By using Mathematica Software<sup>®</sup> [11], one can see that inequality (2.10) is not true for  $x > 2.284$ . Similarly, the reverse inequality of (2.10) is not true, e.g. for  $x < 2.2$ . These show that,  $I$  and  $S(A, G)$  cannot be compared to each other. In order to prove inequality (1.24), we will use the following result proved in [20]: The inequality

$$(2.11) \quad S > Q$$

holds true. By writing (2.11) as  $S(a, b) > Q(a, b)$  for  $a = Q$ ,  $b = G$ , and remarking that  $Q(a, b) = \sqrt{(a^2 + b^2)/2}$  and that  $(Q^2 + G^2)/2 = A^2$ , we get the first inequality of (1.24). The second inequality is well known (see [15] for history and references).

By using  $I(a, b) < A(a, b) = (a + b)/2$  for  $a = Q$  and  $b = G$  we get  $I(Q, G) < (Q + G)/2$ . On the other hand by inequality  $(a + b)/2 < \sqrt{(a^2 + b^2)/2}$  and  $(Q^2 + G^2)/2 = A^2$ , inequality (1.25) follows as well. This completes the proof.  $\square$

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BABEŞ-BOLYAI UNIVERSITY DEPARTMENT OF MATHEMATICS STR. KOGALNICEANU NR. 1  
400084 CLUJ-NAPOCA, ROMANIA

*E-mail address:* jsandor@math.ubbcluj.ro

DEPARTMENT OF MATHEMATICAL INFORMATION TECHNOLOGY, UNIVERSITY OF JYVÄSKYLÄ,  
40014 JYVÄSKYLÄ, FINLAND,

*E-mail address:* bhayo.barkat@gmail.com